# DECOMPOSITIONS OF WEIGHTED CONDITIONAL EXPECTATION TYPE OPERATORS

Y. ESTAREMI

Abstract. In this paper we investigate boundedness, polar decomposition and spectral decomposition of weighted conditional expectation type operators on $L^2(\Sigma)$.

## 1. Introduction and Preliminaries

Let $(X, \Sigma, \mu)$ be a complete $\sigma$-finite measure space. For any sub-$\sigma$-finite algebra $\mathcal{A} \subseteq \Sigma$ with $1 \leq p \leq \infty$, the $L^p$-space $L^p(X, \mathcal{A}, \mu_{|\mathcal{A}})$ is abbreviated by $L^p(\mathcal{A})$, and its norm is denoted by $\|.\|_p$. All comparisons between two functions or two sets are to be interpreted as holding up to a $\mu$-null set. The support of a measurable function $f$ is defined as $S(f) = \{x \in X; f(x) \neq 0\}$. We denote the vector space of all equivalence classes of almost everywhere finite valued measurable functions on $X$ by $L^0(\Sigma)$.

For a sub-$\sigma$-finite algebra $\mathcal{A} \subseteq \Sigma$, the conditional expectation operator associated with $\mathcal{A}$ is the mapping $f \to E^{\mathcal{A}} f$, defined for all non-negative, measurable function $f$ as well as for all $f \in L^p(\Sigma)$, $1 \leq p \leq \infty$, where $E^{\mathcal{A}} f$, by the Radon-Nikodym theorem, is the unique $\mathcal{A}$-measurable function satisfying

$$\int_A f d\mu = \int_A E^{\mathcal{A}} f d\mu, \quad \forall A \in \mathcal{A}.$$

As an operator on $L^p(\Sigma)$, $E^{\mathcal{A}}$ is idempotent and $E^{\mathcal{A}}(L^p(\Sigma)) = L^p(\mathcal{A})$. If there is no possibility of confusion, we write $E(f)$ in place of $E^{\mathcal{A}}(f)$. This operator will play a major role in our work and we list here some of its useful properties:

- If $g$ is $\mathcal{A}$-measurable, then $E(fg) = E(f)g$.
- $|E(f)|^p \leq E(|f|^p)$.
- If $f \geq 0$, then $E(f) \geq 0$; if $f > 0$, then $E(f) > 0$.
- $|E(fg)| \leq E(|f|^p)|^{\frac{1}{p}} E(|g|^q)|^{\frac{1}{q}}$, where $\frac{1}{p} + \frac{1}{q} = 1$ (Hölder inequality).
- For each $f \geq 0$, $S(f) \subseteq S(E(f))$.

A detailed discussion and verification of most of these properties may be found in [11]. We recall that an $\mathcal{A}$-atom of the measure $\mu$ is an element $A \in \mathcal{A}$ with $\mu(A) > 0$ such that for each $F \in \mathcal{A}$, if $F \subseteq A$, then either $\mu(F) = 0$ or $\mu(F) = \mu(A)$. A measure space $(X, \Sigma, \mu)$ with no atoms is called a non-atomic measure space. It is well-known fact that every $\sigma$-finite measure space $(X, \mathcal{A}, \mu_{|\mathcal{A}})$ can be partitioned

2000 *Mathematics Subject Classification.* 47B47.
*Key words and phrases.* Conditional expectation, Polar decomposition, Spectral decomposition, Aluthge transformation.

uniquely as $X = \left(\bigcup_{n\in\mathbb{N}} A_n\right) \cup B$, where $\{A_n\}_{n\in\mathbb{N}}$ is a countable collection of pairwise disjoint $\mathcal{A}$-atoms and $B$, being disjoint from each $A_n$, is non-atomic (see [12]).

Combinations of conditional expectation operators and multiplication operators appear often in the study of other operators such as multiplication operators, weighted composition operators and integral operators. Specifically, in [9], S.-T. C. Moy characterized all operators on $L^p$ of the form $f \to E(fg)$ for $g$ in $L^q$ with $E(|g|)$ bounded. Eleven years later, R. G. Douglas, [6], analyzed positive projections on $L^1$ and many of his characterizations are in terms of combinations of multiplications and conditional expectations. More recently, P.G. Dodds, C.B. Huijsmans and B. De Pagter, [3], extended these characterizations to the setting of function ideals and vector lattices. J. Herron presented some assertions about the operator $EM_u$ on $L^p$ spaces in [7, 8].
In [4, 5] we investigated some classic properties of multiplication conditional expectation operators $M_wEM_u$ on $L^p$ spaces. Let $f \in L^0(\Sigma)$, then $f$ is said to be conditionable with respect to $E$ if $f \in \mathcal{D}(E) := \{g \in L^0(\Sigma) : E(|g|) \in L^0(\mathcal{A})\}$. Throughout this paper we take $u$ and $w$ in $\mathcal{D}(E)$. In this paper we present some results on the boundedness, polar decomposition and spectral decomposition of this operators in $L^2(\Sigma)$, using different methods than those employed in [5].

## 2. Polar decomposition

**Theorem 2.1.** The operator $T = M_wEM_u : L^2(\Sigma) \to L^2(\Sigma)$ is bounded if and only if $(E(|w|^2)^{\frac{1}{2}})(E(|u|^2)^{\frac{1}{2}}) \in L^\infty(\mathcal{A})$ and in this case $\|T\| = \|(E(|w|^2)^{\frac{1}{2}})(E(|u|^2)^{\frac{1}{2}})\|_\infty$.

**Proof** Suppose that $(E(|w|^2)^{\frac{1}{2}})(E(|u|^2)^{\frac{1}{2}}) \in L^\infty(A)$. Let $f \in L^2(\Sigma)$. Then

$$\|T(f)\|_2^2 = \int_X |wE(uf)|^2 d\mu = \int_X E(|w|^2)|E(uf)|^2 d\mu \leq \int_X E(|w|^2)E(|u|^2)E(|f|^2)|d\mu.$$

Since $|E(uf)| \leq (E(|u|^2)^{\frac{1}{2}}(E(|f|^2)^{\frac{1}{2}}$. Thus

$$\|T\| \leq \|E(|w|^2)^{\frac{1}{2}}(E(|u|^2))^{\frac{1}{2}}\|_\infty.$$

To prove the converse, let $T$ be bounded on $L^2(\Sigma)$ and consider the case that $\mu(X) < \infty$. Then for all $f \in L^2(\Sigma)$ we have

$$\begin{aligned}\|T(f)\|_2^2 = \int_X |wE(uf)|^2 d\mu &= \int_X E(|w|^2)|E(uf)|^2 d\mu \\ &\leq \|T\|^2 \int_X |f|^2 d\mu.\end{aligned}$$

For each $n \in \mathbb{N}$, define

$$E_n = \{x \in X : |u(x)|(E(|w|^2))^{\frac{1}{2}}(x) \leq n\}.$$

Each $E_n$ is $\Sigma$-measurable and $E_n \uparrow X$. Define $G_n = E_n \cap S$ for each $n \in \mathbb{N}$, where $S = S(|u|(E(|w|^2))^{\frac{1}{2}})$. Let $A \in \mathcal{A}$ and define

$$f_n = \bar{u}(E(|w|^2))^{\frac{1}{2}}\chi_{G_n \cap A}$$

for each $n \in \mathbb{N}$. It is clear that $f_n \in L^{\infty}(\Sigma)$ for all $n$ (which in our case implies $f_n \in L^2(\Sigma)$). For each $n$,

$$\begin{aligned}\|T(f_n)\|_2^2 &= \int_X E(|w|^2)|E(uf_n)|^2 d\mu \\ &= \int_X (E(|w|^2))^2 (E(|u|^2\chi_{G_n}.\chi_A))^2 d\mu \\ &= \int_A [E(|w|^2)E(|u|^2\chi_{G_n})]^2 d\mu \\ &\leq \|T\|^2 \int_X |f_n|^2 d\mu \\ &= \|T\|^2 \int_A E(|u|^2\chi_{G_n})E(|w|^2) d\mu.\end{aligned}$$

Since $A$ is an arbitrary $\mathcal{A}$-measurable set and the integrands are both $\mathcal{A}$-measurable functions, we have

$$[E(|w|^2)E(|u|^2\chi_{G_n})]^2 \leq \|T\|^2 E(|u|^2\chi_{G_n})E(|w|^2)$$

almost everywhere. That is

$$[E(((E(|w|^2))^{\frac{1}{2}}|u|\chi_{E_n})^2\chi_S)]^2 \leq \|T\|^2 E((|u|\chi_{E_n}(E(|w|^2))^{\frac{1}{2}})^q\chi_S).$$

Since

$$S = \sigma(|u|(E(|w|^2))^{\frac{1}{2}}) = \sigma|u|^2E(|w|^2)$$

and

$$|u|^2E(|w|^2)\chi_S = |u|^2E(|w|^2),$$

we have

$$E(((E(|w|^2))^{\frac{1}{2}}|u|\chi_{E_n})^2\chi_S) \leq \|T\|^2.$$

Thus

$$(E|w|^2)^{\frac{1}{2}}(E|u|^2\chi_{E_n})^{\frac{1}{2}} \leq \|T\|.$$

This implies that $(E|w|^2)^{\frac{1}{2}}(E|u|^2\chi_{E_n})^{\frac{1}{2}} \in L^{\infty}(\mathcal{A})$ and

$$\|(E|w|^2)^{\frac{1}{2}}(E|u|^2)^{\frac{1}{2}}\|_{\infty} \leq \|T\|.$$

Moreover, since $E_m \uparrow X$, the conditional expectation version of the monotone convergence theorem implies $\|(E(|w|^2))^{\frac{1}{2}}(E(|u|^2))^{\frac{1}{2}}\|_{\infty} \leq \|T\|$. □

**proposition 2.2.** Let $g \in L^{\infty}(\mathcal{A})$ and let $T = M_wEM_u : L^2(\Sigma) \to L^2(\Sigma)$ be bounded. If $M_gT = 0$, then $g = 0$ on $\sigma(E(|w|^2)E(|u|^2))$.

**Proof.** Let $f \in L^2(\Sigma)$. Then $gwE(uf) = M_gT(f) = 0$. Now, by Theorem 2.1

$$0 = \|M_gT\|^2 = \||g|^2E(|w|^2)E(|u|^2)\|\infty,$$

which implies that $|g|^2E(|w|^2)E(|u|^2) = 0$, and so $g = 0$ on $\sigma(E(|w|^2)E(|u|^2))$. □

**Theorem 2.3.** The bounded operator $T = M_wEM_u$ is a partial isometry if and only if $E(|w|^2)E(|u|^2) = \chi_A$ for some $A \in \mathcal{A}$.

**Proof.** Suppose $T$ is partial isometry. Then $TT^*T = T$, that is $Tf = E(|w|^2)E(|u|^2)Tf$, and hence $(E(|w|^2)E(|u|^2) - 1)Tf = 0$ for all $f \in L^2(\Sigma)$. Put $S = S(E(|u|^2))$ and

$G = S(E(|w|^2))$. By Proposition 2.2. we get that $E(|w|^2)E(|u|^2) = 1$ on $S \cap G$, which implies that $E(|w|^2)E(|u|^2) = \chi_A$, where $A = S \cap G$.

Conversely, suppose that $E(|w|^2)E(|u|^2) = \chi_A$ for some $A \in \mathcal{A}$. It follows that $A = S \cap G$, and we have

$$TT^*T(f) = E(|w|^2)E(|u|^2)Tf = \chi_{S\cap G} wE(uf) = wE(uf),$$

where we have used the fact that $S(Tf) = S(|Tf|^2) \subseteq S \cap G$, which this is a consequence of Hölder's inequality for conditional expectation $E$. $\square$

The spectrum of an operator $A$ is the set

$$\sigma(A) = \{\lambda \in \mathbb{C} : A - \lambda I \quad \textit{is not invertible}\}.$$

It is well known that any bounded operator $A$ on a Hilbert space $\mathcal{H}$ can be expressed in terms of its polar decomposition: $A = UP$, where $U$ is a partial isometry and $P$ is a positive operator. (An operator is positive if $\langle Pf, f\rangle \geq 0$, for all $f \in \mathcal{H}$.) This representation is unique under the condition that $kerU = kerP = kerA$. Moreover, $P = |A| = (A^*A)^{\frac{1}{2}}$.

Let $q(z)$ be a polynomial with complex coefficients: $q(z) = \sum_{n=0}^{N} \alpha_n z^n$. If $T$ is a bounded operator on $L^2(\Sigma)$, then the operator $q(T)$ is defined by $q(z) = \alpha_0 I + \sum_{n=1}^{N} \alpha_n T^n$. Let $M_\varphi$ be a bounded multiplication operator on $L^2(\Sigma)$, then $q(M_\varphi)$ is also bounded and $q(M_\varphi) = M_{q\circ\varphi}$. By the continuous functional calculus, for any $f \in C(\sigma(M_\varphi))$, we have $g(M_\varphi) = M_{g\circ\varphi}$.

**Proposition 2.4.** Let $S = S(E(|u|^2))$ and $G = S(E(|w|^2))$. If $f \in C(\sigma(M_{E(|u|^2)}))$ and $g \in C(\sigma(M_{E(|w|^2)}))$, Then

$$f(T^*T) = f(0)I + M_{(E(|u|^2))^{-1}.\chi_S} \left(M_{f\circ(E(|u|^2)E(|w|2))} - f(0)I\right) M_{\bar{u}}EM_u$$

and

$$g(TT^*) = g(0)I + M_{(E(|w|^2))^{-1}.\chi_G} \left(M_{g\circ(E(|u|^2)E(|w|2))} - g(0)I\right) M_w EM_{\bar{w}}.$$

**Proof.** For all $f \in L^2(\Sigma)$, $T^*T(f) = \bar{u}E(|w|^2)E(uf)$ and $TT^*(f) = wE(|u|^2)E(\bar{w}f)$. By induction, for each $n \in \mathbb{N}$,

$$(T^*T)^n(f) = \bar{u}(E(|w|^2))^n(E(|u|^2))^{n-1}E(uf), \quad (TT^*)^n(f) = w(E(|u|^2))^n(E(|w|^2))^{n-1}E(\bar{w}f).$$

So

$$q(T^*T) = q(0)I + M_{(E(|u|^2))^{-1}.\chi_S} \left(M_{q\circ(E(|u|^2)E(|w|2))} - q(0)I\right) M_{\bar{u}}EM_u$$

and

$$q(TT^*) = q(0)I + M_{(E(|w|^2))^{-1}.\chi_G} \left(M_{q\circ(E(|u|^2)E(|w|2))} - q(0)I\right) M_w EM_{\bar{w}}.$$

By the Weierstrass approximation theorem we conclude that, for every $f \in C(\sigma(M_{E(|u|^2)}))$ and $g \in C(\sigma(M_{E(|w|^2)}))$,

$$f(T^*T) = f(0)I + M_{(E(|u|^2))^{-1}.\chi_S} \left(M_{f\circ(E(|u|^2)E(|w|2))} - f(0)I\right) M_{\bar{u}}EM_u$$

and

$$g(TT^*) = g(0)I + M_{(E(|w|^2))^{-1}.\chi_G} \left(M_{g\circ(E(|u|^2)E(|w|2))} - g(0)I\right) M_w EM_{\bar{w}}.$$

**Theorem 2.5.** The unique polar decomposition of $T = M_w EM_u$ is $U|T|$, where

$$|T|(f) = \left(\frac{E(|w|^2)}{E(|u|^2)}\right)^{\frac{1}{2}} \chi_S \bar{u} E(uf), \qquad U(f) = \left(\frac{\chi_{S\cap G}}{E(|w|^2)E(|u|^2)}\right)^{\frac{1}{2}} wE(uf),$$

for all $f \in L^2(\Sigma)$.

**Proof.** By Proposition 2.4 we have

$$|T|(f) = (T^*T)^{\frac{1}{2}}(f) = \left(\frac{E(|w|^2)}{E(|u|^2)}\right)^{\frac{1}{2}} \chi_S \bar{u} E(uf).$$

Define a linear operator $U$ whose action is given by

$$U(f) = \left(\frac{\chi_{S\cap G}}{E(|w|^2)E(|u|^2)}\right)^{\frac{1}{2}} wE(uf), \qquad f \in L^2(\Sigma).$$

Then $T = U|T|$ and by Theorem 2.3, $U$ is a partial isometry. Also, it is easy to see that $\mathcal{N}(T) = \mathcal{N}(U)$. Since for all $f \in L^2(\Sigma)$, $\|Tf\|_2 = \|\ |T|f\|_2$, hence $\mathcal{N}(|T|) = \mathcal{N}(U)$ and so this decomposition is unique. $\square$

**Theorem 2.6.** The Aluthge transformation of $T = M_w E M_u$ is

$$\widehat{T}(f) = \frac{\chi_S E(uw)}{E(|u|^2)} \bar{u} E(uf), \qquad f \in L^2(\Sigma).$$

**Proof.** Define operator $V$ on $L^2(\Sigma)$ as

$$Vf = \left(\frac{E(|w|^2)}{(E(|u|^2))^3}\right)^{\frac{1}{4}} \chi_S \bar{u} E(uf), \qquad f \in L^2(\Sigma).$$

Then we have $V^2 = |T|$ and so by direct computation we obtain

$$\widehat{T}(f) = |T|^{\frac{1}{2}} U |T|^{\frac{1}{2}}(f) = \frac{\chi_S E(uw)}{E(|u|^2)} \bar{u} E(uf).$$

$\square$

## 3. Spectral decomposition

The normal operators form one of the best understood and most tractable of classes of operators. The principal reason for this is the spectral theorem, a powerful structure theorem that answers many (not all) questions about these operators. In this section we explore spectral measure and spectral decomposition corresponding to a normal weighted conditional expectation operator $EM_u$ on $L^2(\Sigma)$.

**Definition 3.1.** If $X$ is a set, $\Sigma$ a $\sigma$-algebra of subsets of $X$ and $H$ a Hilbert space, a spectral measure for $(X, \Sigma, H)$ is a function $\mathcal{E} : \Sigma \to B(H)$ having the following properties.
(a) $\mathcal{E}(S)$ is a projection.
(b) $\mathcal{E}(\emptyset) = 0$ and $\mathcal{E}(X) = I$.
(c) If $S_1, S_2 \in \Sigma$. $\mathcal{E}(S_1 \cap S_2) = \mathcal{E}(S_1)\mathcal{E}(S_2)$.
(d) If $\{S_n\}_{n=0}^{\infty}$ is a sequence of pairwise disjoint sets in $\Sigma$, then

$$\mathcal{E}(\cup_{n=0}^{\infty} S_n) = \Sigma_{n=0}^{\infty} \mathcal{E}(S_n).$$

The spectral theorem says that: For every normal operator $T$ on a Hilbert space $H$, there is a unique spectral measure $\mathcal{E}$ relative to $(\sigma(T), H)$ such that $T = \int_{\sigma(T)} z d\mathcal{E}$, where $z$ is the inclusion map of $\sigma(T)$ in $\mathbb{C}$.

J. Herron showed that $\sigma(EM_u) = ess\ range(Eu) \cup \{0\}$, [8]. Also, He has proved that: $EM_u$ is normal if and only if $u \in L^{\infty}(\mathcal{A})$. If $T = EM_u$ is normal, then $T^n = M_{u^n}E$ and $(T^*)^n = M_{\bar{u}^n}E$. So $(T^*)^n T^m = M_{(\bar{u})^n u^m}E$ and

$$P(T, T^*) = \sum_{n,m=0}^{N,M} \alpha_{n,m} T^m (T^*)^n = \sum_{n,m=0}^{N,M} \alpha_{n,m} \bar{u}^n u^m E = P(u, \bar{u})E = EP(u, \bar{u}).$$

Where $p(z,t) = \sum_{n,m=0}^{N,M} \alpha_{n,m} z^m t^n$. If $q(z) = \sum_{n=0}^{N} \alpha_n z^n$, then $q(T) = \sum_{n=0}^{N} \alpha_n u^n E$. Hence by the Weierstrass approximation theorem we have $f(T) = M_{f(u)}E$, for all $f \in C(\sigma(EM_u))$. Thus $\phi : C(\sigma(EM_u)) \to C^*(EM_u, I)$, by $\phi(f) = M_{f(u)}E$, is a unital $*$-homomorphism. Moreover, by Theorem 2.1.13 of [10], $\phi$ is also a uniqe $*$-isomorphism such that $\phi(z) = EM_u$, where $z : \sigma(EM_u) \to \mathbb{C}$ is the inclusion map.

If $EM_u$ is normal and compact, then $\sigma(EM_u) = \{0\} \cup \{\lambda_n\}_{n\in\mathbb{N}}$ where $\lambda_n \neq 0$ for all $n \in \mathbb{N}$. So, for each $n \in \mathbb{N}$

$$\begin{aligned} E_n &= \{0 \neq f \in L^2(\Sigma) : E(uf) = \lambda_n f\} \\ &= \{0 \neq f \in L^2(\mathcal{A}) : uf = \lambda_n f\} \\ &= L^2(A_n, \mathcal{A}_n, \mu_n), \end{aligned}$$

where $A_n = \{x \in X : u(x) = \lambda_n\}$, $E_0 = \{f \in L^2(\Sigma) : E(uf) = 0\}$, $\mathcal{A}_n = \{A_n \cap B : B \in \mathcal{A}\}$ and $\mu_n \equiv \mu\mid_{\mathcal{A}_n}$. It is clear that for all $n, m \in \mathbb{N} \cup \{0\}$, $E_n \cap E_m = \emptyset$. This implies that the spectral decomposition of $EM_u$ is as follows:

$$EM_u = \sum_{n=0}^{\infty} \lambda_n P_{E_n},$$

where $P_{E_n}$ is the orthogonal projection onto $E_n$. Since $EM_u$ is normal, then $\sigma(EM_u) = ess\ range(u) \cup \{0\}$. So $\{\lambda_n\}_{n=0}^{\infty}$ is a resolution of the identity on $X$.

Suppose that $W = \{u \in L^0(\Sigma) : E(|u|^2) \in L^{\infty}(\mathcal{A})\}$. If we set $\|u\| = \|(E(|u|^2))^{\frac{1}{2}}\|_{\infty}$, then $W$ is a complete $*$-subalgebra of $L^{\infty}(\Sigma)$.

In the sequel we assume that, $\varphi : X \to X$ is nonsingular transformation i.e, the measure $\mu \circ \varphi^{-1}$ is absolutely continuous with respect to the measure $\mu$, and $\varphi^{-1}(\Sigma)$ is a sub-$\sigma$-finite algebra of $\Sigma$. Put $h = d\mu \circ \varphi^{-1}/d\mu$ and $E^{\varphi} = E^{\varphi^{-1}(\Sigma)}$. For $S \in \Sigma$, let $\mathcal{E}(S) : L^2(\Sigma) \to L^2(\Sigma)$ be defined by

$$\mathcal{E}(S)(f) = E^{\varphi} M_{\chi_{\varphi^{-1}(S)}}(f),$$

i.e,

$$\mathcal{E}(S) = E^{\varphi} M_{\chi_{\varphi^{-1}(S)}}.$$

$\mathcal{E}$ defines a spectral measure for $(X, \Sigma, L^2(\mu))$. If $E^{\varphi}M_u$ is normal on $L^2(\Sigma)$, then by Theorem 2.5.5 of [10], $\mathcal{E}$ is the unique spectral measure corresponding to $*$-homomorphism $\phi$ that is defined as follows:

$$\phi : C(\sigma(E^{\varphi}M_u)) \to C^*(EM_u, I), \quad \phi(f) = E^{\varphi} M_{f(u)}.$$

So, for all $f \in C(\sigma(E^{\varphi}M_u))$ we have

$$\phi(f) = \int_X f d\mathcal{E}.$$

In [1] it is explored that which sub-$\sigma$-algebras of $\Sigma$ are of the form $\varphi^{-1}(\Sigma)$ for some nonsingular transformation $\varphi : X \to X$. These observations establish the following theorem.

**Theorem 3.2.** Let $(X, \Sigma, \mu)$ be a $\sigma$-finite measure space, $\varphi : X \to X$ be a nonsingular transformation and let $u$ be in $L^{\infty}(\varphi^{-1}(\Sigma))$. Consider the operator $E^{\varphi}M_u$ on $L^2(\Sigma)$. Then the set function $\mathcal{E}$ that is defined as: $\mathcal{E}(S) = E^{\varphi}M_{\chi_{\varphi^{-1}(S)}}$ for $S \in \Sigma$, is a spectral measure. Also, $\mathcal{E}$ has compact support and

$$E^{\varphi}M_u = \int z d\mathcal{E}.$$

## References

[1] J. T. Campbell, A. Lambert and B. M. Weinstock, Localizing Sets and the Structure of Sigma-Algebras, *Indiana University Mathematics Journal* **47** (1998) 913-938.
[2] P.G. Dodds, C.B. Huijsmans and B. De Pagter, characterizations of conditional expectation-type operators, *Pacific J. Math* **141** (1990) 55-77.
[3] P.G. Dodds, C.B. Huijsmans and B. De Pagter, characterizations of conditional expectation-type operators, Pacific J. Math. **141**(1) (1990), 55-77.
[4] Y. Estaremi, Essential norm of weighted conditional type operators on $L^p$-spaces, *to appear in positivity.*
[5] Y. Estaremi and M.R. Jabbarzadeh, Weighted lambert type operators on $L^p$-spaces, Oper. Matrices **1** (2013), 101-116..
[6] R. G. Douglas, Contractive projections on an $L_1$ space, Pacific J. Math. **15** (1965), 443-462.
[7] J. Herron, Weighted conditional expectation operators on $L^p$ spaces, UNC Charlotte Doctoral Dissertation, 2004.
[8] J. Herron, Weighted conditional expectation operators, Oper. Matrices **1** (2011), 107-118.
[9] Shu-Teh Chen, Moy, Characterizations of conditional expectation as a transformation on function spaces, Pacific J. Math. **4** (1954), 47-63
[10] G. J. Murphy, $C^*$-algebras and operator theory, Academic Press, Boston San Diego, 1990.
[11] M. M. Rao, Conditional measure and applications, Marcel Dekker, New York, 1993.
[12] A. C. Zaanen, Integration, 2nd ed., North-Holland, Amsterdam, 1967.

Y. Estaremi
*E-mail address*: estaremi@gmail.com - yestaremi@pnu.ac.ir

Department of Mathematics, University of Payame noor, p. o. box: 19395-3697, Tehran, Iran.